Model Order Reduction of Large Circuits Using Balanced Truncation

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Abstract- A method is introduced for model order reduction of large circuits extracted from layout. The algorithm, which is based on balanced realization, can be used for reducing the order of circuits before circuit-level simulation. In contrast to Pade-based algorithms which match the reduced order system with original system in some given frequencies, balanced realization based model algorithms provide a nearly optimal matching over all frequencies. Hence the balanced realization method produces stable and more accurate results compared to the Pade-based algorithms for model reduction. In addition given an upper bound for error, it is possible to compute the minimum degree for the reduced order model a priori. A numerically efficient method for balanced truncation of large circuits using the Arnoldi algorithm is presented and experimental results are reported.

I. Introduction

As the minimum feature size in VLSI chips reaches 0.1 micrometers and the frequency reaches 1GHz, the interconnections become the dominant determinants of performance. Under these conditions, an accurate circuit simulation is needed to provide the performance characteristics of the circuit. Since the frequency is high, the interconnections should be modeled as distributed elements rather than lumped elements. Because of the circuit density and features of deep sub-micron process and interconnect technology, many parasitic resistors, capacitors and inductors should be considered. This causes the extracted circuit to have an extremely large number of linear elements. No computer program can perform simulation for such high dimensions in a reasonable amount of time. This creates the need for methods to change the given networks to simpler networks so that we can simulate them.

The first work in this area is AWE [1], which is a method for approximating the transfer function with a reduced order method using the Pade approximation. There exist methods that enable AWE to handle large dimension [2][3][4][5]. A disadvantage of these algorithms is that starting with the system equations, we cannot a priori predict the degree of the reduced system.

The problem of model order reduction has been studied in control engineering [6][7][8] where effective methods have been developed. One such method is *model reduction using balanced realization, which* attempts to minimize the difference of the reduced order model with the original model over all frequencies. In this method we are guaranteed to obtain a stable reduced-order system given a stable initial system. Furthermore, we can obtain a bound on the error over all frequencies and it can be shown that the solution is optimal. This method can be used for both single-input, single-output systems and multi-input, multi-output systems. Silveira [9] used this method for modeling a transmission line. In this paper, we extend his work to model reduction of very large linear circuits. Also we use a provably more efficient technique for achieving the balanced realization.

We introduce the state space form and the system norms in section II. In section III we present balanced realization and model reduction using balanced realization. Section IV gives a fast efficient numerical method for solving the required equations using the Krylov subspace methods. Experimental results are given in section V.

II. State Space Form and System Norms

In this section we introduce the definitions and mathematical tools used in this paper. Using any circuit equation formulation method such as the modified nodal analysis, sparse tableau, etc., a lumped linear, time invariant system can be described by the following system of first order differential equations [10]: $F\dot{\vec{x}} = -M\vec{x} + E\vec{u}$, $\vec{y} = C\vec{x} + D\vec{u}$.

Here vector $\vec{x}$ represents the capacitor voltages and inductor currents in the circuit, matrix M represents the contribution of memory-less elements such as resistors, matrix F represents the contribution from memory elements such as capacitors and inductors, $\vec{y}$ is the output vector, and $\vec{u}$ is the input vector to the network. The $\vec{x}$ vector is called the *state vector*. Matrix E represents how the energy is coupled from the inputs to the states whereas matrix C shows how the states are related to the outputs. We also have matrix D that is due to terms that directly couple energy from input to output. The given equation can be manipulated to: $\dot{\vec{x}} = A\vec{x} + B\vec{u}$, $\vec{y} = C\vec{x} + D\vec{u}$.

This is the *standard state space formulation* for a linear system. Given the state space matrix (A,B,C,D), the transfer function of the system can be written as: $G(s) = C(sI - A)^{-1}B + D$.

Consider that we have a direct term *D*. Without loss of generality, we set this term to zero. After we reduce the order of the system we simply add the *D* term to the resultant system. The system, which has zero *D* term, is called a *strictly proper* system.

To obtain a measure of how far two linear systems are from each other, we need a method to calculate the difference between the outputs of the systems for arbitrary input signals. First we define the l_p norm of the vector signal $\vec{z}(t)$ as [11]:

$$\left\|\vec{z}(t)\right\|_p = \sqrt[p]{\sum_i \int_{-\infty}^{\infty} |z_i(\tau)|^p \, d\tau}$$

Consider that if we let *p=2*, the given norm is the Euclidean norm of the energy of the given vector. Based on the assumption we have on the input signals, different system norms can be defined as explained next.

A. H_∞ *System Norm*

Theorem (SVD) [11]: Any (complex) $l \times m$ matrix A may be factorized into a singular value decomposition $A = U\sum V^H$ where the $l \times l$ matrix U and the $m \times m$ matrix V are unitary, and the $l \times m$ matrix $\sum$ contains a diagonal matrix $\sum_1$ of real, non negative singular values, σ_i , arranged in a decreasing order as in:

$$\sum = \begin{cases} \begin{bmatrix} \Sigma_1 \\ 0 \end{bmatrix} & , l > m \\ \Sigma_1 & , l = m \\ \begin{bmatrix} \Sigma_1 & 0 \end{bmatrix} & , l < m \end{cases}$$

where $\Sigma_1 = diag\{\sigma_1, \sigma_2, \sigma_3, ..., \sigma_k\}$; $k = \min(l, m)$ and $\sigma_1 \geq \sigma_2 \geq .... \geq \sigma_k$ and the singular values σ_i are positive square roots of the k largest eigen-values of both AA^H and $A^H A$: $\sigma_i(A) = \sqrt{\lambda_i(A^H A)} = \sqrt{\lambda_i(AA^H)}$. Matrices U and V are unit eigen-vectors of AA^H and $A^H A$, respectively.

Now consider a strictly proper linear stable system G(s). The H_∞ norm of a system is defined as [11]:

$$\|G(s)\|_\infty = \max_\omega \sigma_1(G(j\omega))$$

In terms of performance we see that H_∞ norm is the peak of the transfer function "magnitude".

The H_∞ norm also has several time domain performance interpretations. For example it can be shown that [11]:

$$\|G(s)\|_\infty = \max_{\vec{u}(t)\neq 0} \frac{\|\vec{y}(t)\|_2}{\|\vec{u}(t)\|_2}$$

From the norms introduced it is evident that for strictly proper systems, if we have a small $\|G_1(s) - G_2(s)\|_\infty$ then the systems are almost identical and we can use $G_2(s)$ instead of $G_1(s)$.

B. Hankel Norm

We start by defining the *Observibility and Controlibility Gramians* [11]. The Controlibility Gramian is a measure of how much the input energy is coupled to the states. The Observibility Gramian is a measure of how the states and the output are coupled to each other. For the linear system given by (A, B, C, D), the Controlibility Gramian is defined as:

$$W_r = \int_0^\infty e^{A\tau} B B^T e^{A^T \tau} d\tau$$

It can be shown that if the eigen-values of the matrix A are to the left of the $j\omega$ axis ($\mathrm{Re}[\lambda_i(A)] < 0, \forall i$) which means that if our system is stable, then the Controlibility Gramian can be obtained by solving the Lyapunov equation:

$$AW_r + W_r A^T + BB^T = 0$$

In the same way, the Observibility Gramian is defined as:

$$W_o = \int_0^\infty e^{A^T \tau} C^T C e^{A\tau} d\tau$$

Again it can be shown that if all the eigen-values of the A matrix are to the left $j\omega$ axis, the Observibility Gramian can be obtained by:

$$A^T W_o + W_o A + C^T C = 0$$

Now the Hankel singular values of a system are defined as:

$$\sigma_i(G(s)) = \sqrt{\lambda_i\ (W_r W_o)}, i = 1,2,\cdots,n \quad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$$

Usually the σ_i's are placed in a matrix as:

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \ddots & \\ 0 & 0 & \sigma_n \end{bmatrix}$$

Hankel singular norm of a system is defined as: $\|G(s)\|_H = \sigma_1$

The time domain performance interpretation of the Hankel norm of a system is [11]:

$$\|G(s)\|_H = \max_{\vec{u}(t)} \frac{\sqrt{\int_0^\infty \|\vec{y}(t)\|_2^2}}{\sqrt{\int_{-\infty}^0 \|\vec{u}(t)\|_2^2}}$$

The Hankel norm may be interpreted as a kind of induced norm from past inputs to future outputs.

III. Model Reduction and Balanced Truncation

In this section the theory of model reduction based on balanced realization is reviewed and two important theorems about the order reduction of linear systems are presented. Consider a strictly proper linear system *(A, B, C)*. Suppose we change the basis of our space by the linear transformation: $\vec{z} = T\vec{x}$

The state space form of the system in the new basis is written as: $\dot{\vec{z}} = \hat{A}\vec{z} + \hat{B}\vec{u}$, $\vec{y} = \hat{C}\vec{z}$ where: $\hat{A} = TAT^{-1}, \hat{B} = TB$, $\hat{C} = CT^{-1}$

It can be shown that the Gramians in this realization can be obtained by [6]: $\hat{W}_r = T^{-1} W_r T^{-T}$ and $\hat{W}_o = T^T W_o T$. It is now clear that by choosing T properly, we can change many properties of Gramains matrices.

If the Observibility and Controlibility Gramians for a realization of a system are equal to each other and to the Σ matrix, then that realization is called a balanced realization. Moore [6] proposed balancing the system and then discarding the states corresponding to small Hankel singular values to obtain a reduced order model. The basic idea behind this method is to equalize the coupling of energy from input to states and from states to outputs, i.e. via the linear transformation T so that the Observibility and Controlibility Gramians become equal. Then states that have small coupling are discarded. The procedure is as follows. First find a transformation so that the system realization becomes balanced. (It is shown that such a realization exists for every system [6].) Then divide the state space form for that realization as:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, C = [C_1 \quad C_2]\ \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$$

$$\Sigma_1 = diag(\sigma_1, \sigma_2, \cdots, \sigma_k), \Sigma_2 = diag(\sigma_{k+1}, \sigma_{k+2}, \cdots, \sigma_n)$$

The reduced order system has a degree of k and is written as: $\dot{\vec{x}} = A_{11}\vec{x} + B_1\vec{u}$, $\vec{y} = C_1\vec{x}$. Also the reduced order transfer function can be written as: $G_a^k = C_1(sI - A_{11})^{-1} B_1$.

Theorem 1 *[8] : Let G(s) be a stable rational transfer function with Hankel singular values $\sigma_1 > \sigma_2 > \cdots > \sigma_n$ where each σ_i has multiplicity r_i and let $G_a^k(s)$ be obtained by truncating the balanced realization of $G(s)$ to the first ($r_1 + r_2 + \cdots + r_k$) states. Then:* $\|G(s) - G_a^k(s)\|_\infty \leq 2(\sigma_{k+1} + \sigma_{k+2} + \cdots + \sigma_n)$

Theorem 2 *[8]: Let G(s) be a stable rational transfer function of degree n with Hankel singular values $\sigma_1 > \sigma_2 > \cdots > \sigma_n$. Let H(s) be any arbitrary stable transfer function of degree $r < n$. It can be shown that:* $\|G(s) - H(s)\|_\infty \geq \sigma_{k+1}$

Theorem 3 *[12] The relative error of reduced-order model is:*

$$\frac{\|C(sI - A)^{-1}B - C_1(sI - A_{11})^{-1}B_1\|}{\|C(sI - A)^{-1}B\|} \leq \frac{2(\sigma_{k+1} + \sigma_{k+2} + \cdots + \sigma_n)}{\sigma_1}$$

After computing the Hankel singular values of a matrix using these theorems, we can find the proper degree for the reduced

system and also the absolute and relative error for this approximation.

IV. Numerical Methods for Large Systems

Safonov and Chiang [7] have shown that a reduced system can be directly obtained without calculating the balanced realization.

Procedure Safonov-Chiang:

Input Data: A,B,C,D,k

1. Compute matrices $V_{R,k}$ *and* $V_{L,k} \in \Re^{n\times k}$ *whose columns form bases for the respective right and left eigen-spaces of* $W_r W_o$ *associated with the "big" eigen-values* $\sigma_1{}^2, \cdots, \sigma_k{}^2$.

2. Let $E = V_{L,k}{}^T V_{R,k}$, *compute its singular value decomposition:* $U_E \Sigma_k V_E{}^T = E$.

3. Let $S_L = V_{L,k} U_E \Sigma^{-1/2} \in \Re^{n\times k}$, $S_R = V_{R,k} V_E \Sigma^{-1/2} \in \Re^{n\times k}$, *compute the state space realization:*

$$\hat{A} = S_L{}^T A S_R \qquad \hat{B} = S_L{}^T B$$
$$\hat{C} = C S_R \qquad \hat{D} = D$$

End Procedure

The k^{th} order approximation to the original transfer function is: $\hat{G}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} + \hat{D}$ which is exactly the same as doing truncation for the balanced realization. The numerical robustness of the given algorithm depends on how the bases $V_{R,k}$ and $V_{L,k}$ are calculated. It is also clear that we still need to calculate W_r and W_o which are the solutions of the Lyapunov equation.

Looking at the procedure, it is clear that dimensions of W_r, W_o, $V_{R,k}$ and $V_{L,k}$ are high. (I.e., equal to the order of the original system). The rest of the algorithm works with reduced order matrices, which can easily be handled. Therefore, we need an algorithm for solving the Lyaponuv equations and an algorithm for obtaining the big eigenvalues and the corresponding eigenvectors of the matrix $W_r W_o$.

Saad [12] proposed a method for solving Large Lyapunov equations. The method is based on solving the Lyapunov equations directly using the formula:

$$W_r = \int_0^\infty e^{A\tau} b b^T e^{A^T \tau} d\tau$$

with a Krylov subspace method. In this approach we need to generate an orthonormal basis : $V_m = [v_1, ..., v_m]$ of the Krylov subspace K_m via the well-known Arnoldi algorithm.

Arnodi Algorithm:

Input Data: A,b

1. Compute $v_1 = b/\|b\|_2$.

For j=1,2,…,m

2a. Compute $w = Av_j$.

2b. Compute coefficients $h_{i,j}$ *such that* $w = w - \sum_{i=1}^{j} h_{i,j} v_i$ *is Orthogonal to all previous* v_i's.

2c. Compute $h_{j+1,j} = \|w\|_2$ *and* $v_{j+1} = w/h_{j+1,j}$.

End Algorithm

Saad Procedure uses the following approximation (with provable error bound): $e^{tA}b \approx \beta V_m e^{tH_m} e_1$ where e_1 is the first column of identity matrix, $\beta = \|b\|_2$, $H_m = [h_{ij}]$. Substituting this approximation into expression for *W* results in:

$$W_m = V_m \left(\int_0^\infty e^{\tau H_m}(\beta e_1)(\beta e_1)^T e^{\tau H_m{}^T} d\tau\right) V_m^T = V_m G_m V_m^T.$$

It is then clear that G_m is the solution to the $m \times m$ Lyapunov equation: $H_m G_m + G_m H_m^T + \beta^2 e_1 e_1{}^T = 0$ which can be solved using the ***Hammaring method*** [13]. The second Lyapunov equation is solved in a similar way.

The above algorithm considers single input, single output circuits. (i.e., B, C are vectors). Using a block Arnoldi algorithm the multi input multi output case can also be solved [14].

In the same paper Safonov and Chiang also proposed a method based on Schur decomposition of $W_r W_o$ for obtaining the $V_{R,k}$ and $V_{L,k}$ after solving Lyaponuv equations. All the eigenvalues and eigenvectors are calculated and the k left and right eigenvectors corresponding to the largest eigenvalues are separated. However, as it is clear from the algorithm, only the largest eigenvalues and their corresponding eigenvectors are needed. Solving the eigenvalue problem for largest and smallest singular values and corresponding left and right eigenvectors for large sparse matrices is a well known problem [15]. We can use an Arnoldi algorithm to get the desired number of eigenvalues and largest right and left eigenvectors. Details are omitted due to space restriction.

The proposed algorithmic flow has been implemented in a program named BTVA (*Balanced Truncation Via Arnoldi*):

Procedure BTVA*:*

Input A,B,C,k

1. Use the Saad procedure to calculate W_r *and* W_o.

2. Use the Krylov subspace methods to calculate eigenvalues and corresponding left and right eigenvectors of $W_r W_o$ *(*$V_{r,k}, V_{l,k}$*).*

3. Choose the degree for reduced order system based on the calculated eigenvalues and desired error bound using Theorems 1 and 3 .

4. Compute the reduced order system $\hat{A}, \hat{B}, \hat{C}$ *using steps 2 and 3 of the Safonov-Chiang Procedure .*

End Procedure

V. Experimental Results

Some examples for reduced circuits using balanced realization are given. First we consider an RLC tree, which is a case frequently encountered in extracted VLSI circuits.

Fig (1) shows the circuit and the response of the circuit by *Hspice* and also shows the response by two reduced order systems. It is clear that reduced system of order four is exactly identical to the real system (It is not distinguishable on the figure from the original response). The second order system has the same delay, but a large error at $t = 0$.

The next circuit is a RC tree network. It is clear that a reduced order system of order 2 is identical to the original system Fig (2). The reduced order 1 system is almost the same but has a large error at time 0. Compared to AWE, the results are better, and in addition, in this method we have guaranteed stability, error bound, and the order of the reduced order system can be predicted ahead based on the error bound we need.

The next example is a multi output, single input system. Fig (3) shows the real response and also the approximated response. By examining the Hankel singular values (cf. Table 1), we conclude that a 2nd order approximation yields very high accuracy.

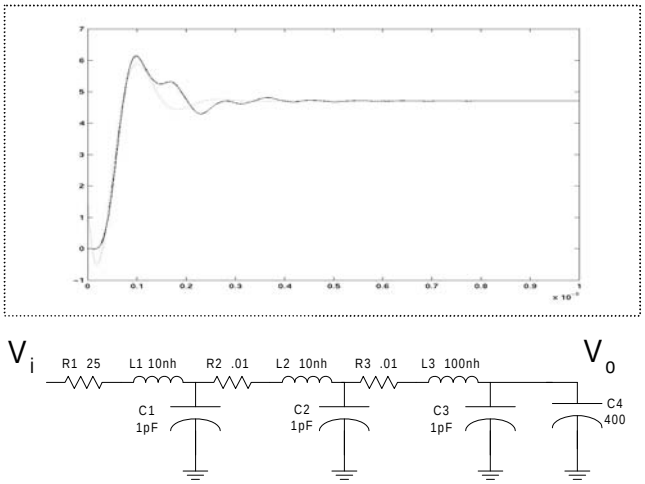

Figure 1 RLC tree with the exact (Solid) second order (dotted) and fourth order (dashed) step response approximation.

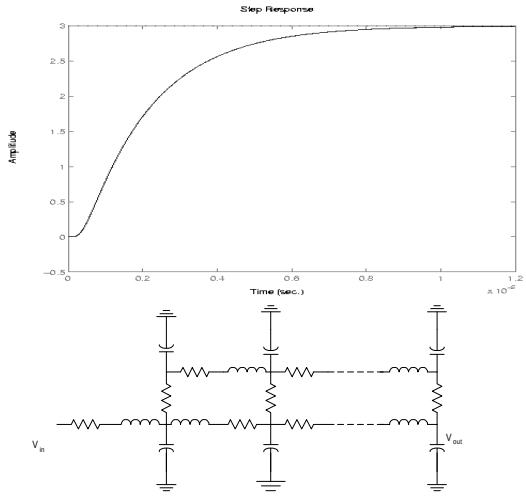

Figure 2 Large RLC circuit with 4th order approximation. The Exact and approximated response are identical.

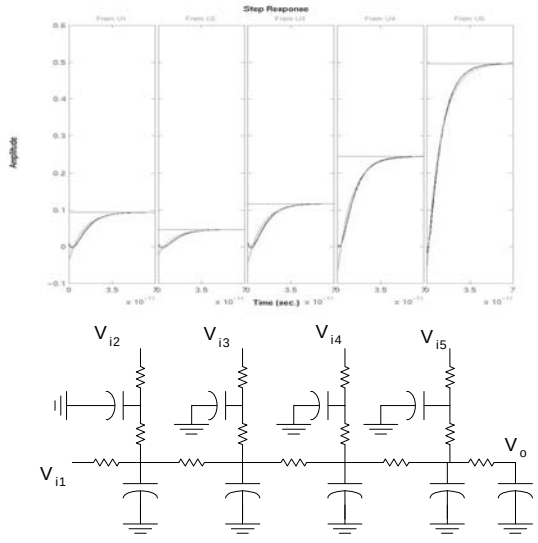

Figure 3 A multi input single output system and its real, first, second order step responses.

1st order	2nd order	Actual	σ
-8.848e+10	-1.0604e+11	-5.362e+11	0.3408
	-2.7188e+11	-5.683e+11	0.0616
		-4.453e+11	0.0175
		-2.688e+11	0.0025
		-9.195e+11	0.0007
		-1.098e+12	0.0001

Table 1 Poles of the actual and reduced-order models of Fig (3) and Hankel singular values of the original system.

VI. Conclusions

The method introduced for approximating the linear parts of a circuit has many advantages over the Pade based methods, which are commonly used for model order reduction. We can get better approximations to a circuit as well as many other beneficial features using balanced realization based method. This method results in very good second order approximation to any RC circuit. For the case of RLC circuits, depending on the number of capacitors and inductors, we usually obtain higher order approximations

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